UV divergence-free QFT on noncommutative plane

2003 J. Phys. A: Math. Gen. 36 L517

(http://iopscience.iop.org/0305-4470/36/39/103)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) L517–L521 PII: S0305-4470(03)64042-4

LETTER TO THE EDITOR

UV divergence-free QFT on noncommutative plane

Anais Smailagic[1] **and Euro Spallucci**[2]

[1] Sezione INFN di Trieste, Strada Costiera 11, 34014 Trieste, Italy
[2] Department of Theoretical Physics, University of Trieste, Strada Costiera 11, 34014 Trieste, Italy

E-mail: anais@ictp.trieste.it and spallucci@trieste.infn.it

Received 27 May 2003
Published 17 September 2003
Online at stacks.iop.org/JPhysA/36/L517

Abstract
We formulate noncommutative qauntum field theory in terms of fields defined as mean value over coherent states of the noncommutative plane. No ∗-product is needed in this formulation and noncommutativity is carried by a modified Fourier transform of fields. As a result the theory is UV finite and the cutoff is provided by the noncommutative parameter θ.

PACS number: 11.10.Nx

Recent revival of interest in noncommutative theories has been triggered by the results in string theory. The early results in this subject [1–4], have been followed by a vast number of papers dealing with the problem of formulating a noncommutative quantum mechanics [6] and field theory. Though it is possible to formally define these models [7] it is hard to perform any calculation directly in terms of noncommutative variables. It has emerged that the most promising approach is to 'simulate' noncommutativity in the space of ordinary functions by the use of ∗-product, as has been attempted long time ago in ordinary quantum mechanics [8]. We have recently shown that the same results can be obtained by suitable redefinition of noncommutative coordinates in terms of canonical ones [9]. In this case, the effect of noncommutativity manifests itself as an 'external, constant magnetic field'.

On the other hand, quantum field theory has been so far formulated only by replacing ordinary commutative product of fields by a ∗-product in the original Lagrangian [4, 7]. This is due to the fact that field theories are formulated in Lagrangian formalism, unlike quantum mechanics which uses Hamiltonian formulation allowing us to work with phase space coordinates. This difference does not allow a straightforward extension of the nice and simple description available in quantum mechanics. In fact, one has to find a self-consistent way of treating noncommutativity of coordinates only.

With the above consideration in mind, we would like to remark that a fundamental effect of noncommutativity is the change in nature of the coordinate space which becomes *blurry* because of the existence of a minimal length determined by the noncommutative parameter θ. In momentum space description of quantum field theory one expects a natural cutoff provided by θ, thus rendering the theory UV finite. Similar ideas, motivated by quantum gravity effects have been introduced in [5]. UV finiteness should be the first test of a successfully formulated noncommutative quantum field theory (NCQFT). Every paper carries this basic expectation, but it has not been achieved if one deals with $*$-product. The reason is that, in order to perform calculations, one is forced to expand in θ the $*$-product and to keep only the first few terms. As has been asserted in many papers, the divergences of Feynman diagrams are not cured by the θ-parameter. Instead of UV finiteness one finds again UV divergences of the commutative field theory. The only effect of noncommutativity in this approach is to generate new, non-planar, Feynman diagrams. Final result is UV/IR mixing which is defying renormalization group expectations.

In this letter, we shall present a way to reformulate NCQFT which incorporates θ as a natural cutoff both in propagators and vertices, in such a way as to render the theory UV finite.

We shall describe a noncommutative scalar field theory on $(2+1)$-dimensional spacetime. We follow usual wisdom to keep time as a commutative coordinate in order to avoid problems with unitarity [10], while space coordinates are noncommutative. We choose to work in a plane being the simplest noncommutative geometry.

The noncommutative plane is described by space coordinates satisfying commutation rules given by

$$[\mathbf{X}^i, \mathbf{X}^j] = \mathrm{i}\theta\epsilon^{ij} \qquad i, j = 1, 2 \tag{1}$$

$$[\mathbf{X}^i, \mathbf{P}_j] = \mathrm{i}\delta^i_j \tag{2}$$

$$[\mathbf{P}_i, \mathbf{P}_j] = 0 \tag{3}$$

where we have chosen units $\hbar = c = 1$. θ has dimensions of a length squared and measures the noncommutativity of coordinates. Conjugate momenta $\mathbf{P}_i$ are chosen to satisfy standard commutation rules. As a consequence of (1), the noncommutative plane is divided into plaquettes of area θ. One cannot speak of points anymore and the space becomes blurry.

This immediately leads to the question: how to define a function (field) of space coordinates?

In our view, the main point in formulating a NCQFT is to find a proper set of states which allow us to define mean value of a function $F(\mathbf{X}^1, \mathbf{X}^2)$. The difference with respect to commutative theory stems from the fact that $\mathbf{X}^1$ and $\mathbf{X}^2$ are operators having *no common position eigenvectors* $|x^1x^2\rangle$ due to (1). In order to look for a convenient set of states let us introduce a set of operators defined as

$$\mathbf{Z} \equiv \frac{1}{\sqrt{2}}(\mathbf{X}^1 + \mathrm{i}\mathbf{X}^2) \tag{4}$$

$$\mathbf{Z}^\dagger \equiv \frac{1}{\sqrt{2}}(\mathbf{X}^1 - \mathrm{i}\mathbf{X}^2). \tag{5}$$

The new operators satisfy commutation relation

$$[\mathbf{Z}, \mathbf{Z}^\dagger] = \theta. \tag{6}$$

One can recognize that the $\mathbf{Z}, \mathbf{Z}^\dagger$ operators satisfy the commutation relation of creation/annihilation operators, of ordinary quantum mechanics, with the formal substitution $\hbar \longrightarrow \theta$. Thus, the commutative limit $\theta \to 0$ corresponds to the classical limit, $\hbar \to 0$, of quantum mechanics. It is known, since the seminal work of Glauber in quantum optics [11], that there exist *coherent states* which are eigenstates of annihilation operator. The advantage of working with operators $\mathbf{Z}, \mathbf{Z}^\dagger$, in place of $\mathbf{X}^1$ and $\mathbf{X}^2$, is that there exist eigenstates satisfying

$$\mathbf{Z}|Z\rangle = z|Z\rangle \tag{7}$$

$$\langle Z|\mathbf{Z}^\dagger = \langle Z|\bar{z} \tag{8}$$

having complex eigenvalues z. The explicit form of the normalized $|Z\rangle$ states is

$$|Z\rangle \equiv \exp\left(-\frac{z\bar{z}}{2\theta}\right)\exp\left(-\frac{z}{\theta}\mathbf{Z}^\dagger\right)|0\rangle \tag{9}$$

where, the vacuum state $|0\rangle$ is annihilated by $\mathbf{Z}$.

The $|\mathbf{Z}\rangle$ states are the coherent states of the noncommutative plane and satisfy the completeness relation

$$\frac{1}{\pi\theta}\int \mathrm{d}z\,\mathrm{d}\bar{z}|z\rangle\langle z| = 1. \tag{10}$$

Coherent states allow us to associate with any operator $F(\mathbf{X}^1, \mathbf{X}^2)$ an ordinary function $F(z)$ as

$$F(z) \equiv \langle z|F(\mathbf{X}^1, \mathbf{X}^2)|z\rangle \tag{11}$$

with the use of the mean value of $\mathbf{X}^1$, $\mathbf{X}^2$ given by

$$\langle z|\mathbf{X}^1|z\rangle = \sqrt{2}\,\mathrm{Re}\,z \tag{12}$$

$$\langle z|\mathbf{X}^2|z\rangle = \sqrt{2}\,\mathrm{Im}\,z. \tag{13}$$

Above definitions open the way for a definition of the quantum fields on the noncommutative plane. Let us first define the noncommutative version of the Fourier transform

$$F(z) = \int \frac{\mathrm{d}^2 p}{2\pi} f(p)\langle z|\exp(\mathrm{i}p_j\mathbf{X}^j)|z\rangle. \tag{14}$$

With the help of (12), (13) the mean value of noncommutative plane wave can be rewritten as

$$\langle z|\exp(\mathrm{i}p_j\mathbf{X}^j)|z\rangle = \langle z|\exp(\mathrm{i}p_+\mathbf{Z}^\dagger)\exp(p_-\mathbf{Z})\exp\left(\frac{p_-p_+}{2}[\mathbf{Z}^\dagger, \mathbf{Z}]\right)|z\rangle \tag{15}$$

where $p_\pm \equiv (p_1 \pm \mathrm{i}p_2)/\sqrt{2}$. We have used the Hausdorff decomposition of the exponent due the noncommutativity of the coordinates which introduces additional factor in the definition of the plane wave on the noncommutative plane. This is the crucial point of our method, i.e. the noncommutativity is seen as a modified Fourier transform of ordinary functions, given by

$$F(z) = \int \frac{\mathrm{d}^2 p}{2\pi} f(p)\exp\left[-\frac{\theta}{4}\left(p_1^2 + p_2^2\right)\right]\exp\left[+\mathrm{i}\frac{p_1}{\sqrt{2}}(z+\bar{z}) + \frac{p_2}{\sqrt{2}}(z-\bar{z})\right]. \tag{16}$$

The above result shows that noncommutativity produces a Gaussian dumping factor. To emphasize the difference between commutative and noncommutative cases, let us choose $f(p) = \mathrm{const}$ corresponding to the maximum spread in momentum. The Fourier transform gives

$$F(z) = \frac{4\pi}{\theta}\exp\left[-\frac{4}{\theta}z\bar{z}\right]. \tag{17}$$

Thus, we find a Gaussian distribution. The reason is that the Gaussian 'remembers' the noncommutativity of the space. Even if the momentum has maximal spread, the uncertainty of the coordinates can shrink only to a minimal width proportional to $\sqrt{\theta}$, indicating blurriness of space. In the commutative limit $\theta \to 0$ one recovers the usual Dirac delta function. As a result of the above discussion, one can assert that the noncommutativity can be introduced in the Fourier transform by replacing ordinary plane waves by Gaussian wavepackets. *A posteriori* this conclusion sounds quite natural. Now we are ready to define a quantum field on a noncommutative plane. A scalar field of mass m will be described through the expansion

$$\phi(t,z) = \sum_{E,p} \left[\mathbf{a}_p^\dagger \exp(-\mathrm{i}Et)\langle z|\exp(\mathrm{i}p_j\mathbf{X}^j)|z\rangle + \text{h.c.}\right] \tag{18}$$

where, $\mathbf{a}_p^\dagger, \mathbf{a}_p$ are usual creation/annihilation operators acting on Fock states with definite energy and momentum. They are the same as in the commutative case since the momenta commute among themselves.

Armed with the above definitions, let us compute the noncommutative version of the Feynman propagator, which is

$$\begin{aligned} G(t_1-t_2, z_1-z_2) &= \langle \vec{p}=\vec{0}|T[\phi(t_1,z_1)\phi(t_2,z_2)]|\vec{p}=\vec{0}\rangle \\ &= \int \frac{\mathrm{d}E}{\sqrt{2\pi}} \exp[-\mathrm{i}E(t_1-t_2)] \int \frac{\mathrm{d}^2p}{2\pi} G(E,\vec{p}^2) \\ &\quad \times \exp\left[\mathrm{i}\frac{p_1}{\sqrt{2}}(z_1+\bar{z}_1-z_2-\bar{z}_2)\right) + \frac{p_2}{\sqrt{2}}(z_1-\bar{z}_1-z_2+\bar{z}_2)\Big] \end{aligned} \tag{19}$$

where the momentum space propagator is

$$G(E,\vec{p}^2) \equiv \frac{1}{-E^2+\vec{p}^2+m^2}\exp\left(-\frac{\theta}{2}\vec{p}^2\right). \tag{20}$$

The above result nicely displays the expected UV cutoff arising from the noncommutativity of the coordinates. Thus, in our approach the effect of noncommutativity that everyone expects is achieved with the help of coherent states. We would like to remark that the modification of the Fourier transform follows from the definition of the mean value over coherent states and is not an ad hoc construction.

Having constructed a dumped Feynman propagator (19), we want to find the corresponding Green function equation. We find it to be

$$\begin{aligned} \left[-\partial_t^2 + \partial_{z_1}\partial_{\bar{z}_1} + m^2\right] G(t_1-t_2, z_1-z_2) &= \delta(t_1-t_2) \\ \times \frac{2\pi}{\theta}\exp\left[-\frac{1}{4\theta}(z_1+\bar{z}_1-z_2-\bar{z}_2)^2 + \frac{1}{4\theta}(z_1-\bar{z}_1-z_2+\bar{z}_2)^2\right] \end{aligned} \tag{21}$$

Again, we find a natural extension in the noncommutative plane, i.e. Dirac delta function of coordinates is replaced by a Gaussian function. The commutative result is recovered as $\theta \to 0$.

Based on previous results, we define the Lagrangian of the noncommutative scalar field as

$$L = \tfrac{1}{2}[(\partial_t\phi)^2 - \partial_{\bar{z}}\phi\partial_z\phi + m^2\phi^2] - V(\phi). \tag{22}$$

It is important to point out that the product of fields in (22) is not a $*$-product, but ordinary product of functions. We do not need a $*$-product since the noncommutativity is embedded in the Fourier transform of a single field (16) rather than in the product among fields. The Feynman rules following from (22) are the standard ones, except that both vertices and propagators are endowed with their own Gaussian dumping factors ensuring UV finiteness of the theory. Localization of noncommutative effects within the Fourier transform of single field avoids unwanted cancellation among Gaussian factors which has taken place when working with $*$-products [12].

Note added in proof. While this paper was in the process of refereeing, another paper of ours, written after this one, was published in [13]. We discuss in [13] the formulation of the Feynman path integral using the coherent state formalism introduced here.

References

[1] Witten E 1996 *Nucl. Phys.* B **460** 335
[2] Seiberg N and Witten E 1999 *J. High Energy Phys.* JHEP09(1999)032
[3] Konechny A and Schwarz A 2002 *Phys. Rep.* **360** 353–465
[4] Douglas M R and Nekrasov N A 2001 *Rev. Mod. Phys.* **73** 977
[5] Padmanabhan T 1987 *Class. Quantum Grav.* **4** L107
Padmanabhan T 1997 *Phys. Rev. Lett.* **78** 1854
[6] Malik R P, Mishra A K and Rajasekaran G 1998 *Int. J. Mod. Phys.* A **13** 4759
Nair V P 2001 *Phys. Lett.* B **505** 249
Nair V P and Polychronakos A P 2001 *Phys. Lett.* B **505** 267
Bellucci S, Nersessian A and Sochichiu C 2001 *Phys. Lett.* B **522** 345
Banerjee R and Kumar S 1999 *Phys. Rev.* D **60** 085005
[7] Alvarez-Gaume L and Wadia S R 2001 *Phys. Lett.* B **501** 319
Alvarez-Gaume L and Barbon J L F 2001 *Int. J. Mod. Phys.* A **16** 1123
[8] Weyl H 1927 *Z. Phys.* **46** 1
Wigner E P 1932 *Phys. Rev.* **40** 749
Moyal G E 1949 *Proc. Camb. Phys. Soc.* **45** 99
[9] Smailagic A and Spallucci E 2002 *J. Phys.* A **35** L363
Smailagic A and Spallucci E 2001 *Phys. Rev.* D **65** 107701
[10] Chaichian M, Demichev A, Presnajder P and Tureanu A 2001 *Eur. Phys. J.* C **20** 767
[11] Glauber R J 1963 *Phys. Rev.* **131** 2766
[12] Chaichian M, Demichev A and Presnajder P 2000 *Nucl. Phys.* B **567** 360